On State-Dependent Streaming Erasure Codes over the Three-Node Relay Network

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***Abstract*—This paper investigates low-latency adaptive streaming codes for a three-node relay network. A source node transmits a sequence of source packets (messages) to the destination through a relay node. We focus on a particular case where the link connecting the source and relay nodes is almost reliable, but the link connecting the relay to the destination is not. The relay node can observe the erasure pattern that has occurred in the transmission between the source node and itself and adapt its relaying strategy based on that observation. Every source packet must be perfectly recovered by the destination with a strict delay T, as long as the number of erasures in the relay-to-destination link lies below some design parameter. We then characterize capacity as a function of such design parameter. The achievability scheme employs two different relaying strategies, based on whether an erasure has or has not occurred in the link from source to relay. The converse is proven by analyzing a periodic erasure pattern and lower bounding the minimum redundancy across channel packets. We show that the achievable rate can be improved compared to non-adaptive schemes previously proposed, indicating that exploiting the knowledge of the erasure pattern by the relay node is essential in achieving capacity.**

I. Introduction

A number of emerging applications including online real-time gaming, real-time video streaming (video conference with multiple users) and healthcare (under the name tactile internet) require efficient low-latency communication. In these applications, data packets are generated at the source in a sequential fashion and must be transmitted to the destination under strict latency constraints. When packets are lost over the network, significant amount of error propagation can occur and suitable methods for error correction are necessary.

Traditionally, packet losses are handled either through automatic repeat request (ARQ) or forward error correction (FEC). For low-latency, long-distance communications, ARQ is not viable due to the round-trip delay being possibly higher than the delay constraint, thus FEC schemes are considered more appropriate candidates. Traditional FEC codes, such as LDPC, however, also introduce large latency due to large blocks, and a new family of codes designed for this strict decoding-delay constraints have been studied in the literature under the name of streaming codes. Previous works have studied particular, useful cases. In [1], the authors studied a point-to-point (i.e., two nodes—source and destination) setting under a maximal burst erasure pattern. In [2], the authors have studied, separately, burst erasures and arbitrary erasures. In [3], the authors have extended the erasure pattern, allowing for both burst erasures and arbitrary erasures. In particular, it was shown that random linear codes [4] are optimal if we are concerned only with correcting arbitrary erasures. Other works that have further studied various aspects of low-latency streaming codes include [5]–[13].

Most of this prior work has focused on a point-to-point communication link. However, many applications can be modeled as a three-node relayed network, involving a relay node between source and destination. This might occur, for example, when two users communicate through a server, or when a user communicates with a server through a nearby gateway that connects the internet to an internal network where the server is connected, e.g. in cloud applications. Motivated by such considerations, streaming codes for such a setting were first introduced in [14], extended to to a multi-hop network in [15] and to an adaptive relay in [16].

In [14], the authors focus on time-invariant (or channel-state-invariant) codes. In particular, the relay does not adapt to the known erasure pattern that has occurred from source to relay, thus the relay must always be prepared for the worst-case scenario, even though it knows the worst-case has not occurred. This was improved in [16], where an adaptive scheme is presented and shown to improve upon the non-adaptive capacity.

In this paper, we focus on a particular scenario where the source-to-relay link is relatively reliable, with at most one erasure happening in rare situations, while the relay-to-destination link is unreliable. This setting is motivated by the downlink in cloud applications, where the first link models an internal and reliable network, and the second link models a possibly unreliable internet connection, including a wireless link. In this setting, we guarantee the recovery of the source packet if and only if the number of erasures occurring in the transmission window is limited by some design parameter η. This is helpful in order to obtain outage probabilities under a probabilistic model, and also allows us to capture worst-case scenarios, which are of interest in low-latency streaming applications [17].

For this setting, we propose an achievable scheme, which although similar to [16], is shown to work under this more restrictive setting that does not allow for error propagation, while the general scheme in [16] does not meet this requirement. More importantly, we derive a tight upper bound, which relies on a novel entropy property that lower bounds the necessary redundancy across channel packets in a streaming setting—something that, to the best of our knowledge, had not been proven previously.

II. Problem Statement

In this paper, we consider a network with one source, one relay and one destination. The source wishes to transmit a sequence of messages $\{s_t\}_{t=0}^{\infty}$ to the destination, through the relay, with a strict delay constraint T. That is, each message s_t should be recovered by the destination by time $t+T$. Furthermore, we assume there is no direct link between source and destination. In this preliminary work, we assume the link from source to relay, denoted "first link", introduces at most $N_1 = 1$ erasure during any window of length $T+1$, that is, any source packet is subject to at most one erasure in the first link from the time it has been generated up to the time it should be recovered by the destination. For the link from relay to destination, we do not assume a maximum number of erasures, however, we guarantee recovery of a source packet s_t if and only if at most η erasures happen from time t up to $t+T$. Note that this condition yields a stronger guarantee of recovery, by reducing error propagation, compared to previous models. That is, we are guaranteed to recover s_t under these conditions even if we have been unable to recover some (or any) $s_{t'}$, $t' < t$.

In the following, we present the formal definitions for the problem. For simplicity, we define $\mathbb{F}_e^n = \mathbb{F}^n \cup \{*\}$.

Definition 1. *An $(n_1, n_2, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

- *One sequence of source messages $\{s_t\}_{t=0}^{t=\infty}$, where $s_t \in \mathbb{F}^k$.*
- *One encoding function $f_t : \underbrace{\mathbb{F}^k \times \cdots \times \mathbb{F}^k}_{t+1 \text{ times}} \to \mathbb{F}^{n_1}$ each used by the source node at time t to generate $x_t^{(1)} = f_t(s_0, s_1, \ldots, s_t)$.*
- *A relaying function $g_t : \underbrace{\mathbb{F}_e^{n_1} \times \cdots \mathbb{F}_e^{n_1}}_{t+1 \text{ times}} \to \mathbb{F}^{n_2}$ used by the relay at time t to generate $x_t^{(2)} = g_t(y_0^{(1)}, y_1^{(1)}, \ldots, y_t^{(1)})$, where $y_t^{(1)}$ is the output of the channel from source to relay at time t, defined in detail in the sequence.*
- *One function: $\varphi_{t+T} = \underbrace{\mathbb{F}_e^{n_2} \times \cdots \times \mathbb{F}_e^{n_2}}_{t+T+1 \text{ times}} \to \mathbb{F}^k$ used by the destination at time $t+T$ to generate an estimate $\hat{s}_t = \varphi_{t+T,i}(y_0^{(2)}, y_1^{(2)}, \ldots, y_{t+T}^{(2)})$, where $y_t^{(2)}$ is the output of the channel from relay to destination at time t, defined in detail in the sequence.*

Definition 2. *An erasure sequence is a binary sequence denoted by $e^{(\ell)} \triangleq \{e_t^{(\ell)}\}_{t=0}^{\infty}$, $\ell = 1, 2$, where $e_t^{(\ell)} = \mathbf{1}\{$an erasure occurs at time t in link $\ell\}$.*

Definition 3. *The mapping $h_n : \mathbb{F}^n \times \{0,1\} \to \mathbb{F}_e^n$ of an erasure channel is defined as $h_n(x, e) = \begin{cases} x, & \text{if } e = 0 \\ *, & \text{if } e = 1 \end{cases}$.*

For any erasure sequences $e^{(1)}$ and $e^{(2)}$ and any $(n_1, n_2, k, T)_{\mathbb{F}}$-streaming code, the following input-output relation holds for each $t \in \mathbb{Z}_+$: $y_t^{(1)} = h_{n_1}(x_t^{(1)}, e_t^{(1)})$, where $e^{(1)}$ is such that $\sum_{t'=t}^{t+T} e_{t'}^{(1)} \leq 1$ for any t. For the second link, the following input-output relation holds for for each $t \in \mathbb{Z}_+$: $y_t^{(2)} = h_{n_2}(x_t^{(2)}, e_t^{(2)})$. As mentioned previously, we make no assumption about $e^{(2)}$ in the channel definition.

Remark 1. *Note that, due to this channel definition, the capacity under a traditional sense is zero, since it is possible that all packets are erased in the second link and therefore no information can be transmitted. Note that the same holds for a probabilistic channel under strict delay constraint - it is possible that all channel packets from transmission up to the recovery deadline are erased and therefore it is impossible to guarantee recovery under a probabilistic model.*

Considering the remark above, we define a (T, N_1, η)-capacity[1] in which we guarantee recovery of a source packet s_t as long as the number of erasures in the second link from time t up to $t+T$ is at most η, which is a design parameter.

Definition 4. *In this work, an $(n_1, n_2, k, T)_{\mathbb{F}}$-streaming code is said to be (T, N_1, η)-achievable if the following holds: $\sum_{t'=t}^{t+T} e_{t'}^{(1)} \leq N_1$ and $\sum_{t'=t}^{t+T} e_{t'}^{(2)} \leq \eta$ implies $\hat{s}_{t+T} = s_t$ for any $s_t \in \mathbb{F}^k$.*

Definition 5. *The rate of an $(n_1, n_2, k, T)_{\mathbb{F}}$-streaming code is $R = \frac{k}{\max(n_1, n_2)}$.*

Definition 6. *The (T, N_1, η)-capacity, denoted by $C_{T,N_1,\eta}$ is the maximum rate achievable by $(n_1, n_2, k, T)_{\mathbb{F}}$-streaming codes that are (T, N_1, η)-achievable.*

As mentioned earlier, in this paper, we focus on the case $N_1 = 1$, which models an almost reliable first link where erasures rarely occur.

In the following sections, we present an example motivating the use of channel-state-dependent codes, and then we present an upper bound and a lower bound on the (T, N_1, η)-capacity and we show that they converge.

III. Motivating Example

As a demonstration of our scheme, let us consider the scenario $N_1 = 1$, $\eta = 2$ and $T = 3$ and compare the time-invariant strategy presented in [14] to the proposed scheme in this paper. For both schemes, we use a code of rate $1/2$ in the source to relay link. In order to make the demonstration easier, we use $k = 2$ and $n_1 = 4$ for both coding schemes. The encoding strategy is presented in Table I. In the example, $s_t[a, b]$ denotes ath and bth symbols from the source packet at time instant t. For example, $s_0[1, 2]$ denotes multiplexing $s_0[1]$ and $s_0[2]$. Intuitively, we can see that the code employed in the first link should be the same to both schemes, as the source node has no information about the channel state and therefore must be prepared to handle the worst case in both links.

[1]This notion is similar to that of an outage capacity, where we guarantee recovery if the channel condition is better than some (designed) threshold. However, in our case, the design parameter is the number of erasures that the code must handle, instead of an error probability or a probabilistic channel condition.

TABLE I: Code employed from source to relay for correcting $N_1 = 1$ erasures

t\symbol	0	1	2	3	4	5	6	7	8
$x_t^{(1)}[1]$	$s_0[1,2]$	$s_1[1,2]$	$s_2[1,2]$	$s_3[1,2]$	$s_4[1,2]$	$s_5[1,2]$	$s_6[1,2]$	$s_7[1,2]$	$s_8[1,2]$
$x_t^{(1)}[2]$		$s_0[1,2]$	$s_1[1,2]$	$s_2[1,2]$	$s_3[1,2]$	$s_4[1,2]$	$s_5[1,2]$	$s_6[1,2]$	$s_7[1,2]$

Now, note that we are guaranteed to recover both source symbols with a delay of at most 1, since we expect at most one erasure in this window due to the reliability of the first link. Using that information, the relaying scheme presented in Table II is proposed in [14]. Note that the rate achieved by this scheme is $1/3$, and, indeed, it is guaranteed to recover any source packet with a delay of at most T under any $\eta = 2$ erasures. Note that the table contains only 3 rows for simplicity, but each channel symbol contains the entropy of two source symbols in this example, that is, we have $n = 6$.

However, let us assume $x_0^{(1)}$ has not been erased in the first link. Then, this relaying scheme is inefficient, as we could start transmitting information about s_0 immediately. With that in mind, consider that an erasure happens only at time $t = 2$. Then, we propose using the relaying strategy presented in Table III. Note that, unlike the previous scheme, we only need 5 symbols in each channel packet, instead of 6. Therefore, we are able to improve the rate from $2/6$ to $2/5$.

Finally, it should be noted that the destination must be made aware whether an erasure has occurred or not, i.e., which "type" of code is being transmitted at each timeslot. This can be easily represented and transmitted with an overhead of at most $\log_2(T+1)$ bits, by transmitting the erasure pattern in the current window of length $(T+1)$. If the size of each source packet is large enough, this overhead is negligible.

IV. Achievability

In this section, we present a general encoding scheme that is $(T, 1, \eta)$-achievable. For simplicity, we start by introducing a scheme which assumes both the relay and the destination know when the erasures have happened in the link from source to relay. Note that the relay easily has access to this information[2], however, the destination must be informed of it by the relay. Then, we upper bound the overhead required to inform the destination and show that it does not scale with the length of the code, therefore, by increasing k, n_1 and n_2, we are able to make this overhead relatively as small as desired. The encoding scheme is identical to the one in [16], however, in the previous paper, the scheme assumes previous packets have been recovered by the destination. In our model, such assumption can not be made, as we are required to recover packets even when previous packets might be lost. We show that, for $N_1 = 1$, this is not an issue and the destination can recover without any knowledge about past packets.

Theorem 1. *Assume the destination is given $e^{(1)}$, that is, the erasure pattern that has occurred in the link from source to relay. Then, there exists an $(n_1, n_2, k, T)_{\mathbb{F}}$-streaming code for which the following holds:*

$$\frac{k}{n_1} = \frac{T-\eta}{T+1-\eta} \quad (1)$$

$$\frac{k}{n_2} = \frac{T+1-\eta}{T+1+\frac{1}{T-\eta}} \quad (2)$$

Proof: In order to prove this theorem, let us consider the following encoding scheme from source to relay: we apply $(T+1-\eta)$ copies of systematic diagonally-interleaved MDS codes with $k' = T-\eta$ and $n_1' = T+1-\eta$. Then, note that, if $x_t^{(1)}$ is not erased, s_t is available entirely at time t at the relay. On the other hand, if $x_t^{(1)}$ is erased, then the relay is able to recover exactly $(T+1-\eta)$ symbols at each subsequent time slot, that is, $(T+1-\eta)$ symbols of s_t are recovered at time $t+1$, another $(T+1-\eta)$ are recovered at time $t+2$, and so on. This property is proven in [14, Lemma 3, p.10]. Further, note that $k = k' \cdot (T+1-\eta)$ and $n_1 = n_1' \cdot (T+1-\eta)$.

Then, the relay employs a different relaying scheme, depending on whether or not $x_t^{(1)}$ has been erased. If $x_t^{(1)}$ has not been erased, we transmit s_t using $(T-\eta)$ copies of MDS block codes with $k'' = T+1-\eta$ and $n_2'' = T+1$. Then, in each time slot from t to $t+T$, there is a contribution of $(T-\eta)$ symbols from $s(t)$, which can be information symbols or parity symbols. An example can be found in Table III, where, for example, $x_3^{(1)}$ has not been erased, thus we transmit s_3 using one copy of an MDS block code with $k'' = 2$ and $n_2'' = 4$.

On the other hand, if $x_t^{(1)}$ has been erased, then we transmit s_t using $(T+1-\eta)$ copies of diagonally-interleaved MDS codes with $k''' = T-\eta$ and $n_2''' = T$, starting at time s_{t+1} and ending at time s_{t+T}. Recall that if $x_t^{(1)}$ has been erased, we recover $(T+1-\eta)$ symbols at each time slot, which is exactly the number of symbols required by the relay at each time instant. After we have recovered all $(T+1-\eta)(T-\eta)$ symbols, the relay is able to generate independent parities and transmit them during the following η time instants. Thus, the relay always has enough information to relay. An example of such code can again be observed in Table III. In this example, $x_2^{(1)}$ has been erased, thus we transmit s_2 through two copies of MDS block codes with $k''' = 1$ and $n_2''' = 3$.

Now, note that any $x_t^{(2)}$ is composed of at most $(T+1-\eta)$ symbols from $s_{t'}$, where an erasure has occurred in the first link at time t' and $t-T \le t' < t$, and other $(T-\eta)(T)$ symbols from non-erased packets. Therefore, $n_2 \le T(T-\eta)+(T+1-\eta)$. Due to the our definition of a relaying function, we then zero-pad $x_t^{(2)}$ in order to obtain $n_2 = T(T-\eta)+(T+1-\eta)$.

Finally, note that in both cases (erased or non-erased), we have $n'' - k'' = \eta$ and $n''' - k''' = \eta$, that is, the code

[2] For example, the relay might have a timeout limit, and considers the packet to be lost if it has not received it by that deadline.

TABLE II: Time-invariant relaying strategy for correcting $\eta = 2$ erasures in the link from relay to destination

t\symbol	0	1	2	3	4	5	6	7	8
$x_t^{(2)}[1]$		$s_0[1,2]$	$s_1[1,2]$	$s_2[1,2]$	$s_3[1,2]$	$s_4[1,2]$	$s_5[1,2]$	$s_6[1,2]$	$s_7[1,2]$
$x_t^{(2)}[2]$			$s_0[1,2]$	$s_1[1,2]$	$s_2[1,2]$	$s_3[1,2]$	$s_4[1,2]$	$s_5[1,2]$	$s_6[1,2]$
$x_t^{(2)}[3]$				$s_0[1,2]$	$s_1[1,2]$	$s_2[1,2]$	$s_3[1,2]$	$s_4[1,2]$	$s_5[1,2]$

TABLE III: Channel-state-dependent relaying strategy for correcting $\eta = 2$ erasures in the link from relay to destination

t\symbol	0	1	2	3	4	5	6	7	8
$x_t^{(2)}[1]$	$s_0[1]$	$s_1[1]$		$s_3[1]$	$s_4[1]$	$s_5[1]$	$s_6[1]$	$s_7[1]$	$s_8[1]$
$x_t^{(2)}[2]$		$s_0[2]$	$s_1[2]$	$s_2[1]$	$s_3[2]$	$s_4[1]$	$s_5[1]$	$s_6[1]$	$s_7[1]$
$x_t^{(2)}[3]$			$s_0[1]+s_0[2]$	$s_1[1]+s_1[2]$	$s_2[1]$	$s_3[1]+s_3[2]$	$s_4[1]+s_4[2]$	$s_5[1]+s_5[2]$	$s_6[1]+s_6[2]$
$x_t^{(2)}[4]$				$s_0[1]+2s_0[2]$	$s_1[1]+2s_1[2]$	$s_2[1]$	$s_3[1]+2s_3[2]$	$s_4[1]+2s_4[2]$	$s_5[1]+2s_5[2]$
$x_t^{(2)}[5]$				$s_2[2]$	$s_2[2]$	$s_2[2]$			

can correctly recover from any η erasures that happen from time t up to $t+T$. Furthermore, since each source packet is transmitted in independent block codes, as can be seen in the previous example, failure to recover a past source packet does not cause error in future packets.

Therefore, this is a $(T, 1, \eta)$-achievable code such that

$$\frac{k}{n_1} = \frac{T-\eta}{T+1-\eta} \tag{3}$$

$$\frac{k}{n_2} = \frac{(T-\eta)(T+1-\eta)}{T(T-\eta)+(T+1-\eta)} = \frac{T+1-\eta}{T+\frac{T+1-\eta}{T-\eta}} \tag{4}$$

which are exactly the desired rates. ■

Proposition 1. *The overhead necessary to convey* $\{e_{t'}^{(1)}\}_{t'=t}^{t+T}$ *from relay to destination is bounded by* $\lceil \log_{|\mathbb{F}|}(T+2) \rceil$.

Proof: Assume an erasure has happened in the first link at time t_ϵ. At each time instant t, we use a naive scheme which transmits the erasure pattern that has occurred from $t-T$ up to t in the first link, that is, $x_t^{(2)}$ contains $\{e_{t'}^{(1)}\}_{t'=t-T}^{t}$. Therefore, the fact that t_ϵ has been erased is conveyed at times $t_\epsilon, t_\epsilon+1, \ldots, t_\epsilon+T$. If any of these channel packets have been successfully recovered by the destination, it is aware that the erasure has occurred at time t_ϵ, thus no other erasures have happened in that window, from the model assumption. Otherwise, all packets have been erased and it is impossible to recover s_{t_ϵ} by the deadline. Finally, note that the binary erasure sequence can easily be represented by $\lceil \log_{|\mathbb{F}|}(T+2) \rceil$ symbols, that is, representing an "1" in any of the $(T+1)$ positions, or only zeros if no erasures have occurred. ■

V. Converse

In order to present the upper bound, we start by presenting the following entropy inequality.

Lemma 1. *Let us denote* $\mathcal{M} = \{1, \ldots, M\}$ *and assume there exists a random variable* s*, and* M *random variables* $x_m, m \in \mathcal{M}$*, such that* $H(s|\{x_m\}_{m\in\mathcal{K}}) = 0$ *holds for any* $\mathcal{K} \subset \mathcal{M}$ *such that* $|\mathcal{K}| = K$*. Then, the following inequality holds:*

$$H(\{x_m\}_{m\in\mathcal{K}}, x_i) \leq \sum_{m\in\mathcal{K}} H(x_m) + H(x_i) - \frac{H(s)}{K} \tag{5}$$

for any $i \in \mathcal{M} \setminus \mathcal{K}$.

Proof of Lemma 1: For simplicity, without loss of generality, let us assume $\mathcal{K} = \{1, 2, \ldots, K\}$ and $i = K+1$. Further, let us define $\mathcal{K}^+ = \mathcal{K} \cup \{K+1\}$. Then, we can write

$$\begin{aligned}
H(\{x_m\}_{m=1}^{K+1}) &\overset{(a)}{=} H(\{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}) \\
&\quad + H(x_{i'}|\{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}) \\
&\overset{(b)}{=} H(\{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}) + H(x_{i'}|\{x_m\}_{m\in\mathcal{K}'}) \\
&\quad - I(x_{i'}; \{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}|\{x_m\}_{m\in\mathcal{K}'}) \\
&\overset{(c)}{\leq} \sum_{m=1}^{K+1} H(x_m) \\
&\quad - I(x_{i'}; \{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}|\{x_m\}_{m\in\mathcal{K}'}) \\
&\overset{(d)}{\leq} \sum_{m=1}^{K+1} H(x_m) - I(x_{i'}; s|\{x_m\}_{m\in\mathcal{K}'})
\end{aligned} \tag{6}$$

where, in (a), $i' \in \mathcal{K}^+$; (b) comes from the relation between mutual information and conditional entropy and the fact that $I(x_{i'}; \{x_m\}_{m\in\mathcal{K}^+\setminus(\{i'\}\cup\mathcal{K}')}|\{x_m\}_{m\in\mathcal{K}'}) = I(x_{i'}; \{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}|\{x_m\}_{m\in\mathcal{K}'})$; (c) comes from conditioning can only reduce entropy; and (d) is due to the fact that $\mathcal{K}^+ \setminus \{i'\}$ is a subset of $\mathcal{M}$ with K elements, therefore, $H(s|\{x_m\}_{m\in\mathcal{K}^+\setminus\{i'\}}) = 0$. Note that this is true for any $\mathcal{K}' \subseteq \mathcal{K}^+ \setminus \{i'\}$. In particular, let us choose $\mathcal{K}' = \{1, 2, \ldots, i'-1\}$, and then let us write the following sum

$$\begin{aligned}
\sum_{i'=1}^{K} H(\{x_m\}_{m=1}^{K+1}) &\leq \sum_{i'=1}^{K} \Bigg(\sum_{m=1}^{K+1} H(x_m) \\
&\qquad - I(x_{i'}; s|\{x_m\}_{m=1}^{i'-1}) \Bigg) \\
&\overset{(a)}{=} K \sum_{m=1}^{K+1} H(x_m) - I(\{x_m\}_{m\in\mathcal{K}}; s) \\
&\overset{(b)}{=} K \sum_{m=1}^{K+1} H(x_m) - H(s)
\end{aligned} \tag{7}$$

where (a) comes from the chain rule of mutual information

TABLE IV: Periodic Erasure Pattern

t	$t+1$	$\cdots$	$t+\eta$	$\cdots$	$t+T$	$t+T+1$	$t+(T+1)+1$	$\cdots$	$t+T+1+\eta$	$\cdots$	$t+z(T+1)$		

and (b) comes from $H(s|\{x_m\}_{m\in\mathcal{K}})=0$. Finally, we have

$$H(\{x_m\}_{m=1}^{K+1}) \leq \sum_{m=1}^{K+1} H(x_m) - \frac{H(s)}{K}$$

which is the desired expression. ∎

Intuitively, this Lemma states that, if a source packet must be recovered from any K packets, then there is a necessary, unavoidable redundancy within $K+1$ packets. We now use this Lemma in order to prove the following Theorem which upper bounds the $(T,1,\eta)$-capacity.

Theorem 2. *The $(T,1,\eta)$-capacity is upper bounded by*

$$C_{T,1,\eta} \leq \min\left(\frac{T-\eta}{T+1-\eta}, \frac{T+1-\eta}{T+1+\frac{1}{T-\eta}}\right). \tag{8}$$

Proof: Consider a periodic erasure pattern containing η erasures in a burst with a period $T+1$, which can be visualized in Table IV. Note that, under this erasure pattern, all source packets must be recovered, since for any window $[t,t+T]$, exactly η erasures have happened. Therefore, we can write the following condition

$$\begin{aligned} &H(\{s_{t'}\}_{t'=t}^{t+\eta}, \{s_{t'}\}_{t'=t+\eta+1}^{t+T+\eta+1} \cdots \{s_{t'}\}_{t'=t+\eta+1+z(T+1)}^{t+\eta+(z+1)(T+1)}) \\ &\quad \leq H(\{x_{t'}^{(2)}\}_{t'=\eta}^{T}, \cdots, \{x_{t'}^{(2)}\}_{t'=\eta+(z+1)(T+1)}^{T+(z+1)(T+1)}). \end{aligned} \tag{9}$$

where $z \in \mathbb{Z}_+$. Now, note that, for any z, we have

$$H(\{x_{t'}^{(2)}\}_{t'=\eta+z(T+1)}^{T+z(T+1)}) \leq \sum_{t'=\eta+z(T+1)}^{T+z(T+1)} H(x_{t'}^{(2)}) - \frac{H(s_{t+z(T+1)})}{T-\eta} \tag{10}$$

due to the following: since there is an erasure at time $t+z(T+1)$ in the source to relay link, $x_{t+z(T+1)}^{(2)}$ can not carry any information about $s_{t+z(T+1)}$ due to causality. Furthermore, instead of erasing $x_{t+z(T+1)}^{(2)}$, we may erase any other channel packet from relay to destination in that window, and $s_{t+z(T+1)}$ must still be recovered from the destination, since only η erasures have occurred. Therefore, among the $(T+1-\eta)$ packets from $\eta+z(T+1)$ to $T+z(T+1)$, any $T-\eta$ packets must be sufficient to recover $s_{t+z(T+1)}$. Then, we apply Lemma 1.

Finally, recall that $H(s_t)=k$ and $H(x_t^{(2)}) \leq n_2$ and let z go from 0 to Z. We then have

$$\begin{aligned} &H\Big(\{s_{t'}\}_{t'=t}^{t+\eta}, \cdots, \{s_{t'}\}_{t'=t+\eta+1+z(T+1)}^{t+\eta+(z+1)(T+1)}, \\ &\qquad \cdots, \{s_{t'}\}_{t'=t+\eta+1+Z(T+1)}^{t+\eta+(Z+1)(T+1)}\Big) \\ &\qquad = (\eta+1)k + (T+1)(Z+1)k \end{aligned} \tag{11}$$

and also

$$\begin{aligned} &H(\{x_{t'}^{(2)}\}_{t'=\eta}^{T}, \cdots, \{x_{t'}^{(2)}\}_{t'=\eta+(Z+1)(T+1)}^{T+(Z+1)(T+1)}) \\ &\qquad \leq (Z+2)(T+1-\eta)n_2 - (Z+2)\frac{k}{T-\eta}. \end{aligned} \tag{12}$$

By making $Z \to \infty$, we have

$$\begin{aligned} &(T+1)k \leq (T+1-\eta)n_2 - \frac{k}{T-\eta} \\ &\implies \frac{k}{n_2} \leq \frac{T+1-\eta}{T+1+\frac{1}{T-\eta}} \end{aligned} \tag{13}$$

Finally, by using the same argument as in [14], we can also show

$$\frac{k}{n_1} \leq \frac{T+1-N_1-\eta}{T+1-\eta}. \tag{14}$$

We do not present the full proof here, but the sketch is as follows: since the source node is unaware of what erasures have happened from relay to destination, it must be able to always handle a burst of η erasures from time $t+T+1-\eta$ up to time $t+T$. Therefore, packet s_t must be recoverable at time $t+T+1-\eta$ at the relay, otherwise, it is impossible for the relay to transmit that information. Then, we bound k/n_1 as a point-to-point channel with effective delay constraint $T+1-\eta$ and N_1 erasures. By applying $N_1=1$, we achieve the desired expression.

To finalize the proof, it suffices to note that

$$\begin{aligned} \frac{k}{\max(n_1,n_2)} &= \min\left(\frac{k}{n_1}, \frac{k}{n_2}\right) && (15) \\ &\leq \min\left(\frac{T-\eta}{T+1-\eta}, \frac{T+1-\eta}{T+1+\frac{1}{T-\eta}}\right) && (16) \end{aligned}$$

which completes the proof. ∎

VI. Conclusion

The paper presents a tight (T,N_1,η)-capacity result for the three-node relayed streaming setting, for the particular scenario where the first link is reliable and introduces rare isolated packet erasures, modeled as at most $N_1=1$ erasures in the first link. Extending the results to an arbitrary $N_1 \geq 1$ is work in progress, and requires extending Lemma 1. We are also working on thoroughly comparing the packet loss rates of our proposed code against ones that allow for error propagation in probabilistic settings, in order to evaluate the impact of this model change.

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